UNIT II

Arithmetic Progression

An ordered set of numbers is called a sequence. The following are some of the examples of sequences.

(1) $1^2, 2^2, 3^2, \ldots\ldots\ldots\ldots\ldots$

(2) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\ldots$

(3) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots\ldots\ldots$

(4) $2, 9, 28, 65, \ldots\ldots\ldots\ldots\ldots$

In a sequence if the first three terms are known, it is possible to write down the succeding terms. In some cases, it may be possible to write down the general term directly. For example, the n^{th} terms of the sequences (1), (2), (3) and (4) are $n^2, \frac{1}{n}, \frac{n}{n+1}, n^3+1$ respectively. A series is the sum of the terms of a sequence. For example, $1^2+2^2+3^2+\ldots\ldots\ldots$ is called a series. In this chapter we consider two important types of series called

(1) Arithmetic series

(2) Geometric series

Arithmetic Progression :

A series, in which the difference of any term and its preceding term is constant is called an Arithmetic series or Arithmetic progression. This constant quantity is called the common difference (C.D.).

Examples :

(1) 1, 4, 7, 10, ...

(2) 6, 1, −4, −9, −14, ...

(3) $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\ldots\ldots\ldots\ldots$

Formula for n^{th} term :

Let us consider the arithmetic progression a, a+d, a+2d,...

For this AP, a is the first term and d is the common difference.

First term $= T_1 = a$

Second term $= T_2 = a+d$

Third term $= T_3 = a+2d$

Fourth term $= T_4 = a+3d$

Proceeding in this way one can easily see that the n^{th} term is given by the formula,

$T_n = a+(n-1)\, d.$

Sum to *n* terms :

Consider the AP; a, a+d, a+2d, ... Let S_n denote the sum to n terms of this A.P.

Then,

$$S_n = a+(a+d)+(a+2d)+\ldots\ldots\ldots+a+(n-1)\, d \qquad \ldots (1)$$

The sum is unaltered if the terms are written from last term to the first term (i.e. in the reverse order).

$\therefore\ S_n = a+(n-1)d+a+(n-2)d+a+(n-3)d+\ldots\ldots+a$ ... (2)

Adding (1) and (2) we get,

$$2S_n = [2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d] + \ldots\ldots\ldots\ldots + [2a+(n-1)d]$$

$$= n[2a+(n-1)d]$$

$$\therefore\ S_n = \frac{n}{2}[2a+(n-1)d]$$

Also $S_n = \frac{n}{2}[a+a+(n-1)d]$

$$= \frac{n}{2}(a+l) \text{ where } l = a+(n-1)d \text{ is the last term.}$$

Note : In an AP, the n^{th} term and the sum to n terms are given by,

$$T_n = a+(n-1)d$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}(a+l)$$

Properties of an AP.

Consider the AP, 3, 5, 7, 9, ... (1)

Multiply each term by 2. We get the series 6, 10, 14, 18, ...

This is also an AP whose first term and common difference are the first term and common difference multiplied by 2.

Divide each term of the AP (1) by 2. We get

$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\ldots\ldots\ldots\ldots\ldots$

This is also an AP. But the first term and common difference are divided by 2.

Add 4 to each term of (1), We get 7, 9, 11, 13, ...

This is also an AP with the same C.D. Subtract 4 from each term of (1); we get $-1, 1, 3, 5,$ .... which is also an AP with the same common difference. Therefore, we note that an AP remains an AP if a constant quantity is added with or subtracted from each term of the AP. It also remains an AP if each term of the AP is multiplied or divided by a constant quantity.

Arithmetic mean (AM) :

x is said to be the AM between a and b if a, x, b are in AP.

$$\therefore \quad x-a = b-x$$

$$\text{(i.e.)} \quad 2x = a+b$$

$$x = \frac{a+b}{2}$$

$x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$ are called the n arithmetic means between a and b if $a, x_1\ x_2, \ldots\ldots\ldots x_n, b$ are in AP. There are $(n+2)$ terms in this AP. Let d be the common difference.

$$\text{Then} \quad b = a+(n+2-1)\,d$$

$$b-a = (n+1)\,d$$

$$d = \frac{b-a}{n+1}$$

$$\therefore \quad x_1 = a+d = a+\frac{b-a}{n+1} = \frac{na+b}{n+1}$$

$$x_2 = a+2d = a+2\,\frac{b-a}{n+1} = \frac{(n-1)\,a+2b}{n+1}$$

$$x_3 = a+3d = a+3\,\frac{b-a}{n+1} = \frac{(n-2)\,a+3b}{n+1}$$

$\vdots$

$$x_n = a+n\,d = a+n\,\frac{(b-a)}{n+1} = \frac{a+n\,b}{n+1}$$

Example 1 :

Find the 40^{th} term of an AP whose 9^{th} term is 465 and 20^{th} term is 388.

Let the AP be a, a+d, a+2d, ...

9^{th} term is 465; $\therefore\ a+8d = 465$ ... (1)

20^{th} term is 388; $\therefore\ a+19d = 388$ ... (2)

Subtracting, $11d = -77$

$\therefore\ d = -7$

From (1), $a-56 = 465$

$a = 465+56$

$= 521$

40^{th} term is $a+39d$

$= 521+39(-7)$

$= 521-273$

$= 248$

Example 2 :

The n^{th} term of a series is $7n-3$. Show that the series is an AP and find the first term and the common difference.

The n^{th} term is $7n-3$

$\therefore\ T_n = 7n-3$ ... (1)

$\therefore$ $(n-1)^{th}$ term is

$$
\begin{aligned}
T_{n-1} &= 7(n-1) - 3 \\
&= 7n - 7 - 3 \\
&= 7n - 10 \\
T_n - T_{n-1} &= 7n - 3 - (7n - 10) \\
&= 7n - 3 - 7n + 10 \\
&= 7 \text{ which is a constant.}
\end{aligned}
$$

$\therefore$ The series is an AP and the common difference is 7.

Put $n = 1$ in (1) $\quad \therefore T_1 = 7 - 3 = 4$

$\therefore$ The first term is 4.

Example 3 :

The rate of monthly salary of a person increases annually in AP. It is known that he was drawing Rs. 200 a month during the 11^{th} year of service and Rs. 380 during the 29^{th} year. Find his starting salary and the rate of annual increments.

Let a be the starting salary and d be the annual increment for each year. Then,

$$a + 10d = 200 \qquad \ldots (1)$$

$$a + 28d = 380 \qquad \ldots (2)$$

Subtracting $\quad 18d = 180$

$$d = 10$$

From (1) $\therefore a + 100 = 200$

$$a = 100$$

$\therefore$ Starting salary $=$ Rs. 100

Annual increment $=$ Rs. 10

Example 4 :

Find three numbers in AP whose sum is 12 and the sum of whose cubes is 408.

Let the three numbers be $a-d$, a, $a+d$.

$$(a-d)+a+(a+d) = 12$$

$$3a = 12 \quad \therefore \quad a = 4$$

Also, $(a-d)^3+a^3+(a+d)^3 = 408$

$$3a^3+6ad^2 = 408$$

$$192+24d^2 = 408$$

$$24d^2 = 216$$

$$d^2 = \frac{216}{24} = 9.$$

$$\therefore d = \pm 3.$$

When $a=4$, $d=3$, the numbers are 1, 4, 7.

When $a=4$, $d=-3$, the numbers are 7, 4, 1.

Example 5 :

Find four integers in AP whose sum is 16 and product is 105.

Let the numbers be $a-3d$, $a-d$, $a+d$, $a+3d$.

$$(a-3d) + (a-d) + (a+d) + (a+3d) = 16.$$

$$4a = 16 \quad \therefore a = 4$$

$$(a-3d)(a-d)(a+d)(a+3d) = 105$$

$$(a^2-9d^2)(a^2-d^2) = 105$$

$$(16-9d^2)(16-d^2) = 105$$

$$256-144d^2-16d^2+9d^4 = 105$$

$$9d^4-160d^2+151 = 0$$

$$9d^4-9d^2-151d^2+151 = 0$$

$$9d^2(d^2-1) -151(d^2-1) = 0$$

$$(d^2-1)(9d^2-151) = 0$$

$$\therefore \quad d^2 = 1 \quad \text{or} \quad d^2 = \frac{151}{9}$$

$$d = \pm 1 \quad \text{or} \quad \pm \sqrt{\frac{151}{9}}$$

When $a=4$, $d=1$, the numbers are 1, 3, 5, 7.

When $a=4$, $d=-1$, the numbers are 7, 5, 3, 1.

$d = \pm \frac{\sqrt{151}}{3}$ is inadmissible since the 4 numbers are not integers in this case.

Example 6 :

Divide 20 into 4 parts which are in AP such that the product of the first and the fourth is to the product of the second and third in the ratio 2 : 3.

Let the 4 parts be $a-3d$, $a-d$, $a+d$, $a+3d$.

$$(a-3d) + (a-d) + (a+d) + (a+3d) = 20$$

$$4a = 20$$

$$a = 5$$

$$(a-3d)(a+3d) : (a-d)(a+d) = 2 : 3$$

$$\frac{a^2-9d^2}{a^2-d^2} = \frac{2}{3}$$

$$3(a^2-9d^2) = 2(a^2-d^2)$$

$$a^2 = 25d^2$$

$$25 = 25d^2$$

$$\therefore d^2 = 1$$

$$d = \pm 1$$

When $a=5$, $d=1$, the 4 parts are 2, 4, 6, 8.

When $a=5$, $d=-1$, the 4 parts are 8, 6, 4, 2.

Example 7 :

A firm produces 1,000 sets of TV during its first year. The total sets produced at the end of 5 years is 7,000. Estimate the annual rate of increase in production, if the increase in each year is uniform. (B.com. Sep. 1985)

First year's production = 1000

Production in 5 years = 7000

Since the rate of increase is uniform, the production in these years forms an AP with

$a = 1000$

$n = 5$

$S_n = 7000$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$7000 = \frac{5}{2}[2000+4d]$$

$$14,000 = 10,000+20d$$

$$4,000 = 20d$$

$$d = \frac{4000}{20} = 200.$$

∴ Annual rate of increase = 200 sets

Example 8 :

Find the sum of all integers between 200 and 500 which are divisible by 7.

The first number between 200 and 500 divisible by 7 is 203, and the last number divisible by 7 is 497.

∴ The series is 203, 210, 217, ... 497.

To find the number of terms in this series we use the formula, $T_n = a + (n-1)\,d$

$$497 = 203 + (n-1)\,7$$
$$= 203 + 7n - 7$$
$$\therefore\ 7n = 497 - 203 + 7$$
$$= 301$$
$$\therefore\ n = \frac{301}{7} = 43$$

$\therefore$ The sum of the series $= \frac{n}{2}(a+l)$

$$= \frac{43}{2}(203+497)$$
$$= \frac{43}{2} \times 700 = 15{,}050.$$

Example 9 :

The sum to n terms of a series is $3n^2+2n$. Show that the series is an AP. Find the first term and the common difference.

Let S_n denote the sum to n terms.

$$S_n = 3n^2 + 2n$$
$$S_{n-1} = 3(n-1)^2 + 2(n-1)$$
$$T_n = S_n - S_{n-1}$$
$$= 3n^2 + 2n - 3n^2 + 6n - 3 - 2n + 2$$
$$= 6n - 1$$
$$T_{n-1} = 6(n-1) - 1$$
$$= 6n - 7$$

$\therefore$ $T_n - T_{n-1} = 6n - 1 - (6n - 7) = 6$ which is a constant.

∴ The series is an AP. The common difference is 6.

Putting $n = 1$ in T_n, we get the first term.

$$T_1 = 6-1 = 5$$

∴ The first term is 5.

Example 10 :

I propose to take 30 consecutive terms of the series $100+99+98+97+\ldots\ldots\ldots\ldots$ At what term must I begin so so that the sum of the series is 1155.

Here we have to find a.

We are given that $n = 30$, $d = -1$, $S_n = 1155$.

$$S_n = \frac{n}{2} [2a+(n-1)d]$$

$$1155 = \frac{30}{2} [2a+(30-1)(-1)]$$

$$1155 = 15(2a-29)$$

$$= 30a-435$$

$$\therefore \quad 30a = 1590$$

$$a = 53$$

Example 11 :

Show that the sum of an AP whose first term is a, second term is b, and the last term is c, is equal to $\frac{(a+c)(b+c-2a)}{2(b-a)}$

First term of the AP $= a$

Second term of the AP $= b$

∴ Common difference $= b-a$

Last term of the AP $= c$

$$T_n = a + (n-1)\,d$$

$$c = a + (n-1)(b-a)$$

$$\frac{c-a}{b-a} = n-1$$

$$\therefore\ n = \frac{c-a}{b-a} + 1$$

$$n = \frac{b+c-2a}{b-a}$$

$$S_n = \frac{n}{2}(a+l)$$

$$= \frac{(b+c-2a)\ (a+c)}{2\,(b-a)}$$

Example 12 :

The sums to n terms of two arithmetic progressions are in the ratio $7n+1 : 4n+2$. Find the ratio of their 11 terms.

Let $a, a+d, a+2d, \ldots\ldots$ and

$a_1, a_1+d_1, a_1+2d_1, \ldots\ldots$ be the two APs.

Given that the ratio of the sums to n terms is $7n+1 : 4n+2$.

$$\therefore\ \frac{\frac{n}{2}[2a+(n-1)\,d]}{\frac{n}{2}[2a_1+(n-1)\,d_1]} = \frac{7n+1}{4n+2}$$

$$\text{(i.e.)}\ \frac{2a+(n-1)\,d}{2a_1+(n-1)\,d_1} = \frac{7n+1}{4n+2} \qquad \ldots (1)$$

The ratio of the 11 terms $= \dfrac{a+10d}{a_1+10d_1}$

$= \dfrac{2a+20d}{2a_1+20d_1}$ [multiply Nr and Dr by 2]

$$= \frac{2a+(21-1)\,d}{2a_1+(21-1)\,d_1} \qquad \dots (2)$$

$$= \frac{7\times 21+1}{4\times 21+2}$$ [comparing the LHS of (1) and (2) we find n=21]

$$= \frac{148}{86} = \frac{74}{43}$$

Example 13 :

If a^2, b^2 c^2 are in AP show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in AP.

Given a^2, b^2, c^2 are in AP.

Adding, $ab+bc+ca$ to each term,

$a^2+ab+bc+ca$, $b^2+ab+bc+ca$, $c^2+ab+bc+ca$ are also in AP.

(i.e.) $(a+b)(a+c)$, $(b+c)(a+b)$, $(c+a)(b+c)$ are in AP.

Divide each term by $(a+b)(b+c)(c+a)$,

(i.e) $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in AP.

Example 14 :

A person is appointed on a basic salary of Rs. 1000 a month and gets an increment of Rs. 50 every year. He contributes 10% of his salary to Provident Fund. What will be total contribution to provident fund during his 25 years of service ?

Basic salary at the time of appointment	=	Rs. 1000
Yearly increment	=	Rs. 50
Period of service	=	25 years.

Total salary for 25 years

$= 1000\times 12+1050\times 12+1100\times 12+ \ldots\ldots\ldots$ 25 terms

$= 12\ [1000+1050+1100+ \ldots\ldots\ldots$ 25 terms$]$

$$= 12\times\frac{25}{2}\ [2\times 1000+(25-1)\ 50]$$

$= 150\ [2000+1200]$

$= 150\times 3200$

$\therefore$ Total contribution to PF in 25 years

$$= 150\times 3200\times\frac{10}{100} = \text{Rs. } 48{,}000$$

Example 15 :

A man repays a loan of Rs. 3250 by paying Rs. 20 in the first month and then increases the payment by Rs. 15 every month. How long will it take to clear his loan?

Payment in the first month = Rs. 20

Payment in the 2nd month = Rs. 35

Payment in the 3rd month = Rs. 50, etc.,

Let the person clear the loan of Rs. 3250 in n monthly instalments.

The payments of 20, 35, 50, 65, ... form an AP.

Then, $S_n = \frac{n}{2}\ [2a+(n-1)\ d]$

$$3250 = \frac{n}{2}\ [2\times 20+(n-1)\ 15]$$

$$6500 = n\ (25+15n)$$

$$\therefore\quad 15n^2+25n-6500 = 0$$

(i.e.) $3n^2+5n-1300 = 0$

$$3n^2-60n+65n-1300 = 0$$

$$3n(n-20) + 65(n-20) = 0$$

$$(n-20)(3n+65) = 0$$

$$n = 20, \text{ or } \frac{-65}{3}$$

$n = \frac{-65}{3}$ is inadmissible since 'n' should be a positive integer.

$\therefore$ The loan will be cleared after 20 months.

Example 16 :

Two posts are offered to a man. In one the starting salary was Rs. 120 per month and the annual increment was Rs. 8; in other the starting salary was Rs. 85 but the annual increment was Rs. 12. The man decided to accept that post which would give him more total earnings in the first twenty years of his service. Which post was acceptable to him? Justify your answer.

In the first post the starting salary was Rs. 120, with an annual increment of Rs. 8.

$\therefore$ The total salary that would have been received in 20 years $= 120\times12+128\times12+136\times12+$ .... This is an AP with common difference 12×8

$$\therefore \quad S_{20} = 12\,[120+128+136+ \text{............ to 20 terms}]$$

$$= 12\times\frac{20}{2}\,[2\times120+(20-1)\,8]$$

$$= 120\,[240+152]$$

$$= \text{Rs. } 47{,}040.$$

The salary that would have been earned in the 2nd post in 20 years

$= 85\times12+97\times12+109\times12+\ldots\ldots\ldots\ldots\ldots$

$= 12\,[\,85+97+109+\ldots\ldots\ldots\ldots\ldots \text{ to } 20 \text{ terms}\,]$

$= 12\times\frac{20}{2}\;[\,2\times85+(20-1)\,12\,]$

$= 120\,[\,170+228\,]$

$= 398\times120$

= Rs. 47,760.

∴ The second post was acceptable.

EXERCISE 6

(1) The 7^{th} term of an AP is 39 and 17^{th} term is 69. Find the series.

(2) The 10^{th} term of an AP is 184 and 16^{th} term is 160. Find the 45^{th} term.

(3) The n^{th} term of a series is $5n-3$. Show that the series is an AP. Find the sum to 20 terms of this AP.

(4) Find the sum of all natural numbers between 200 and 500 which are divisible by 17.

(5) Find the sum of all three digit numbers which are multiples of 13.

(6) Find the sum of all natural numbers from 250 to 600 which are not divisible by 17.

(7) Find three numbers in AP whose sum is 24 and whose product is 440.

Geometric Progression

A series in which the ratio of any term to its preceding term is a constant, is called a Geometric Progression (G.P.) The constant quantity is called the common ratio.

Examples :

(1) 2, 4, 8, 16, ...

(2) 4, 12, 36, 108, ...

(3) a, ar, ar^2, ...

If the first term and the common ratio are known, we can write down the geometric series.

Formula for n^{th} term :

Let a be the first term and r be the common ratio of a GP.

Then the series is a, ar, ar^2, ...

1st term $= T_1 = a$

2nd term $= T_2 = ar$

3rd term $= T_3 = ar^2$

4th term $= T_4 = ar^3$

Proceeding in this way we get the n^{th} term as $T_n = ar^{n-1}$

Formula for sum to n terms:

Let a be the first term and r be the common ratio of a GP. Let S_n denote the sum to n terms.

Then,

$$S_n = a+ar+ar^2+ \ldots\ldots\ldots\ldots +ar^{n-1} \quad \ldots (1)$$

Multiplying both sides by r,

$$r\,S_n = ar+ar^2+ar^3+ \ldots\ldots\ldots\ldots ar^{n-1}+ar^n \quad \ldots (2)$$

Subtracting (2) from (1),

$$S_n(1-r) = a-ar^n$$

$$= a\,(1-r^n)$$

$$\therefore \quad S_n = \frac{a\,(1-r^n)}{(1-r)} \quad \ldots (3)$$

$$= \frac{a\,(r^n-1)}{r-1} \quad \ldots (4)$$

Formula (3) is used if the common ratio is less than 1 and formula (4) is used if the commn ratio is greater than one.

Sum to infinity :

If $|r| < 1$, as $n \to \infty$, $r^n \to 0$.

$$\therefore \quad S_\infty = \frac{a}{1-r}$$

If $|r| > 1$ $\quad S_\infty$ does not exist.

Geometric Mean (GM)

x is said to be the GM between a and b if a, x, b are in GP.

$$\therefore \quad \frac{x}{a} = \frac{b}{x} \quad \text{(ie) } a^2=ab$$

n geometric mean between a and b :

$x_1, x_2, x_3, \ldots\ldots\ldots x_n$ are said to be the GMs between a and b if a, $x_1, x_2, \ldots\ldots\ldots x_n$, b are in GP. There are (n+2) terms in this GP. Let r be the common ratio.

$$\therefore \quad b = ar^{n+2-1}$$

$$\frac{b}{a} = r^{n+1}$$

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$x_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$x_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$x_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

$$\vdots$$

$$x_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Note :

The product of these GMs is

$$x_1, x_2 \ldots\ldots\ldots x_n = a^n\left(\frac{b}{a}\right)^{\frac{1+2+\cdots\cdots\cdots+n}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$= a^n \cdot \frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}}}$$

$$= a^{\frac{n}{2}} \cdot b^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}}$$

Example 1 :

Find three numbers in GP whose sum is 21 and whose product is 216.

Let the three numbers be $\frac{a}{r}$, a, ar

$$\text{Then} \quad \frac{a}{r}+a+ar = 21 \qquad \dots (1)$$

$$\frac{a}{r} \cdot a \cdot ar = 216 \qquad \dots (2)$$

$$a^3 = 216$$

$$\therefore \quad a = 6$$

(1) becomes $\frac{6}{r}+6+6r = 21$

$$6r^2+6r+6 = 21r$$

$$6r^2-15r+6 = 0$$

$$6r^2-12r-3r+6=0$$

$$6r(r-2)-3(r-2)=0$$

$$(r-2)(6r-3)=0$$

$$\therefore \quad r=2 \text{ or } \frac{1}{2}$$

When $a=6$, $r=2$, the numbers are $\frac{6}{2}$, 6, 6×2

(ie) 3, 6, 12

When $a=6$, $r=\frac{1}{2}$, the numbers are 12, 6, 3

Example 2 :

Find the sum to n terms of the series 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, ...

This series is a GP with $a=3$, $r=\frac{2}{3}$

$$S_n=\frac{a(1-r^n)}{1-r}$$

$$=\frac{3\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}}$$

$$=3\times3\left[1-\left(\frac{2}{3}\right)^n\right]=9\left[1-\left(\frac{2}{3}\right)^n\right]$$

Example 3 :

Find the sum to n terms of the series

5++55+555+.................

$$S_n = 5+55+555+\ldots\ldots\ldots\ldots\ldots$$

$$= 5[\,1+11+111+\ldots\ldots\ldots\,]$$

$$= \frac{5}{9}\,[\,9+99+999+\ldots\ldots\ldots\,]$$

$$= \frac{5}{9}\,[\,(10-1)+(100-1)+(1000-1)+\ldots\ldots\ldots\,]$$

$$= \frac{5}{9}\,[\,(10+100+1000+\ldots\ldots\ldots \text{ n terms})-(1+1+1+\ldots\ldots\ldots \text{ to n terms})\,]$$

$$= \frac{5}{9}\left[\frac{10\,(10^n-1)}{10-1} - n\right]$$

$$= \frac{5}{9}\left[\frac{10\,(10^n-1)}{9} - n\right]$$

Example 4 :

Find sum to n terms of the series $\cdot 7+\cdot 77\cdot 777+\ldots\ldots\ldots\ldots$

$$S_n = \cdot 7+\cdot 77\cdot 777+\ldots\ldots\ldots\ldots\ldots$$

$$= 7[\,\cdot 1+\cdot 11+\cdot 111+\ldots\ldots\ldots\ldots\ldots\,]$$

$$= 7\left[\frac{1}{10}+\frac{11}{100}+\frac{111}{1000}+\ldots\ldots\ldots\right]$$

$$= \frac{7}{9}\left[\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\ldots\ldots\ldots\right]$$

$$= \frac{7}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots\ldots\right]$$

$$= \frac{7}{9}\left[n-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots\ldots\ldots\right)\right]$$

$$= \frac{7}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}} \right]$$

$$= \frac{7}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{\frac{9}{10}} \right]$$

$$= \frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right) \right]$$

Example 5 :

Find the least value of n such that the sum to n terms of the series

$1+3+3^2+3^3+$ .... exceeds 3,000.

Given $1+3+3^2+3^3+$ .... to n terms > 3000

(i.e.) $\frac{1\,(3^n-1)}{3-1} > 3000$

$3^n - 1 > 6000$

$3^n > 6001$

Taking logarithm,

$n \log 3 > \log 6001$

$n > \frac{\log 6001}{\log 3}$

$n > \frac{3 \cdot 7783}{0 \cdot 4771}$

$n > 7 \cdot 199$

$\therefore$ The least value of n such that $s_n > 3000$ is 8.

Example 6 :

If a, b, c, d are in GP show that

$$(b-c)^2+(c-a)^2+(d-b)^2 = (a-d)^2$$

Taking $a = a$

$b = ar$

$c = ar^2$

$d = ar^3$

$$LHS = (ar-ar^2)^2 + (ar^2-a)^2 + (ar^3-ar)^2$$

$$= (a^2r^2+a^2r^4-2a^2r^3) + (a^2r^4+a^2-2a^2r^2)$$

$$+ (a^2r^6+a^2r^2-2a^2r^4)$$

$$= a^2r^6-2a^2r^3+a^2$$

$$= (a-ar^3)^2$$

$$= (a-d)^2 = RHS.$$

Example 7 :

Three numbers whose sum is 18 are in AP. If 2, 4, 11 are added to them respectively the resulting numbers are in G.P. Determine the numbers.

Let the three numbers be $a-d$, a, $a+d$.

Then $(a-d)+ a + (a+d) = 18$

$$3a = 18$$

$$a = 6$$

$\therefore$ The numbers are $6-d$, 6, $6+d$.

When 2, 4, 11 are added respectively to these numbers we get the numbers as $8-d$, 10, $17+d$ which are given to be in GP.

$\therefore$ $10^2 = (8-d)(17+d)$

$$100 = 136-9d-d^2$$

(ie) $d^2+9d-36 = 0$

$(d-3)(d+12) = 0$

$\therefore\ d = 3$ or -12

$\therefore$ The numbers are 3, 6, 9, or 18, 6, -6.

Example 8 :

The sum of the digits of a three digit number is 12. The digits are in arithmetic progression. If the digits are reversed, then the number is diminished by 396. Find the numbers.

Let the digits be $a-d$, a, $a+d$.

$\therefore\ (a-d)+a+a+d = 12$

$3a = 12$

$a = 4$

Also given,

$$100(a-d)+10a+a+d = 100(a+d)+10a+(a-d)+396$$

(ie) $100a-100d+10a+a+d - 100a-100d-10a-a+d = 396$

$-198d = 396$

$d = -2$

$\therefore$ The digits are 6, 4, 2

The number is 642.

Example 9 :

The sum of the first three terms of two series, one in an AP and the other in a GP is the same. If the first term of each of these is $\frac{2}{3}$ and the common difference of the AP is equal to the common ratio of the GP find the sum of each series to 20 terms.

Let the AP be a, a+d, a+2d, ...

Then the GP is a, ad, ad^2, ...

$$a = \frac{2}{3}$$

Also, $a + (a+d) + (a+2d) = a+ad+ad^2$

$$\therefore \quad 2a+3d = ad+ad^2$$

$$\frac{4}{3} + 3d = \frac{2}{3}d + \frac{2}{3}d^2$$

$$4+9d = 2d+2d^2$$

$$2d^2-7d-4 = 0$$

$$2d^2-8d+d-4 = 0$$

$$2d(d-4) + 1(d-4) = 0$$

$$(d-4)(2d+1) = 0$$

$$d = 4 \text{ or } -\frac{1}{2}$$

When $a = \frac{2}{3}$, $d = 4$ the sum of 20 terms of the AP is

$$S_{20} = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{20}{2}\left[\frac{4}{3} + 19 \times 4\right]$$

$$= \frac{10 \times 232}{3} = \frac{2320}{3} = 773\tfrac{1}{3}$$

When $a = \frac{2}{3}$, $d = -\frac{1}{2}$,

$$S_{20} = \frac{20}{2}\left[\frac{4}{3} + 19\left(-\frac{1}{2}\right)\right]$$

$$= 10 \times \left(\frac{-49}{6}\right) = \frac{-490}{6} = -81\tfrac{2}{3}$$

When $a = \frac{2}{3}$, $d = 4$, the sum to 20 terms of the GP is

$$S_{20} = \frac{a\,(d^n - 1)}{d-1}$$

$$= \frac{2}{3}\left[\frac{4^{20}-1}{4-1}\right]$$

$$= \frac{2}{9}\,(4^{20}-1)$$

When $a = \frac{2}{3}$, $d = -\frac{1}{2}$

$$S_{20} = \frac{a\,[1-d^n]}{1-d}$$

$$= \frac{\frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^{20}\right]}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{2}{3} \times \frac{2}{3}\left[1-\frac{1}{2^{20}}\right]$$

$$= \frac{4}{9}\left[1-\frac{1}{2^{20}}\right]$$

Example 10 :

If S_1, S_2, S_3 be respectively the sum of n, 2n, 3n terms of a GP, prove that $S_1(S_3-S_2) = (S_2-S_1)^2$

Let the GP be a, ar ar^2, ...

$$S_1 = \frac{a(1-r^n)}{1-r}$$

$$S_2 = \frac{a(1-r^{2n})}{1-r}$$

$$S_3 = \frac{a(1-r^{3n})}{1-r}$$

$$S_1(S_3-S_2) = \frac{a(1-r^n)}{1-r}\left[\frac{a(1-r^{3n})}{1-r} - \frac{a(1-r^{2n})}{1-r}\right]$$

$$= \frac{a^2(1-r^n)}{(1-r)^2}[1-r^{3n}-1+r^{2n}]$$

$$= \frac{a^2(1-r^n)}{(1-r)^2}(r^{2n}-r^{3n})$$

$$= \frac{a^2(1-r^n) \cdot r^{2n}(1-r^n)}{(1-r)^2}$$

$$= \frac{a^2 r^{2n}(1-r^n)^2}{(1-r)^2} \qquad \ldots (1)$$

$$(S_2-S_1)^2 = \left[\frac{a(1-r^{2n})}{1-r} - \frac{a(1-r^n)}{1-r}\right]^2$$

$$= \left(\frac{a}{1-r}\right)^2 [1-r^{2n}-1+r^n]^2$$

$$= \frac{a^2}{(1-r)^2} \cdot [r^n(1-r^n)]^2$$

$$= \frac{a^2 r^{2n} \cdot (1-r^n)^2}{(1-r)^2} \qquad \ldots (2)$$

$\therefore$ From (1) and (2) $S_1(S_3-S_2) = (S_2-S_1)^2$

Example 11 :

If $x = 1+a+a^2+\ldots\ldots\ldots\ldots\infty$

$y = 1+b+b^2+\ldots\ldots\ldots\ldots\infty$

Prove that $1+ab+a^2b^2+\ldots\ldots\ldots\ \infty = \frac{xy}{x+y-1}$ where $|a|$ and $|b|$ are less than 1.

If $|r|<1$, the sum to infinity of the GP $a+ar+ar^2+\ldots\ldots\ldots$ is $\frac{a}{1-r}$

$$\therefore\ x = 1+a+a^2+\ldots\ldots\ldots\ \infty$$

$$= \frac{1}{1-a} \qquad \ldots (1)$$

$$y = 1+b+b^2+\ldots\ldots\ldots\ \infty$$

$$= \frac{1}{1-b} \qquad \ldots (2)$$

Since $|a|<1$, $|b|<1$ we have $|ab|<1$.

$$\therefore\ 1+ab+b^2a^2+\ldots\ldots\ldots\ \infty$$

$$= \frac{1}{1-ab} \qquad \ldots (3)$$

Also
$$\frac{xy}{x+y-1} = \frac{\frac{1}{1-a}\cdot\frac{1}{1-b}}{\frac{1}{1-a}+\frac{1}{1-b}-1}$$

$$= \frac{1}{1-b+1-a-(1-a)(1-b)}$$

$$= \frac{1}{1-ab} \qquad \ldots (4)$$

From (3) and (4),

$$1+ab+a^2 b^2 + \dots\dots\dots\dots \infty = \frac{xy}{x+y-1}$$

Example 12 :

If a, b, c form an AP and b, c, a form a GP show that $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ form an AP.

Since a, b, c are in AP, $2b=a+c$ ... (1)

Since b, c, a are in GP, $c^2=ab$ ... (2)

$$\text{Now } \frac{1}{c} + \frac{1}{b} = \frac{b+c}{bc}$$

$$= \frac{ab+ac}{abc}$$

$$= \frac{c^2+ac}{abc}$$

$$= \frac{c\ (c+a)}{abc}$$

$$= \frac{2bc}{abc} = \frac{2}{a}$$

$\therefore\ \frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ are in AP.

Example 13 :

Three integers form an increasing geometric progression. If the third number is decreased by 16, we get an arithmetic progression. If then the second number is decreased by 2, we again get a GP. Find the numbers.

Let the three integers in GP be a, ar, ar^2

Then a, ar, ar^2-16 are in AP.

$\therefore \quad 2ar = a + ar^2 - 16 \qquad \ldots (1)$

Also a, ar−2, ar²−16, are in GP

$\therefore \quad (ar-2)^2 = a\,(ar^2-16)$

$a^2r^2 - 4ar + 4 = a^2r^2 - 16a$

$1 - ar = -4a$

$$a = \frac{1}{r-4} \qquad \ldots (2)$$

Substituting (2) in (1),

$$\frac{2r}{r-4} = \frac{1}{r-4} + \frac{r^2}{r-4} - 16$$

$2r = 1 + r^2 - 16\,(r-4)$

(ie) $r^2 - 18r + 65 = 0$

$(r-5)\,(r-13) = 0$

$r = 5$ or 13.

When $r = 5$, from (2) $a = 1$

When $r = 13$, $a = \frac{1}{9}$

When $r = 5$, $a = 1$, the numbers are 1, 5, 25.

$a = \frac{1}{9}$ is inadmissible, since the numbers in this case are not integers.

Example 14 :

A person is entitled to receive an annual payment which for each year is less by one–tenth of what it was for the previous year. If the first payment is Rs. 100 show that he cannot receive more than Rs. 1,000 however long he may live.

Payment for the first year = Rs. 100

Payment for the 2nd year $= 100 \times \frac{9}{10}$

Payment for the 3rd year $= 100 \times \left(\frac{9}{10}\right)^2$

Total amount received if he continues to receive for ever

$$= 100 + 100 \times \left(\frac{9}{10}\right) + 100 \times \left(\frac{9}{10}\right)^2 + \dots\dots\dots\dots \infty$$

$$= \frac{100}{1 - \frac{9}{10}} \quad \therefore \left(S_\infty = \frac{a}{1-r} \right)$$

= Rs. 1,000

∴ He cannot receive more than Rs. 1,000.

Example 15:

A man borrows Rs. 5,115 to be paid in 10 monthly instalments. If each instalment is double the value of the last, find the value of the first and the last instalments.

Let the instalments be a, 2a, 4a, 8a, ...

This is a GP with first term a and common ratio 2.

$$S_{10} = \frac{a\,(2^{10} - 1)}{2 - 1}$$

$$5115 = a\,(1024 - 1)$$

$$a = \frac{5115}{1023} = 5$$

$$t_{10} = ar^9 = 5 \cdot 2^9$$
$$= 5 \times 512 = 2560,$$

∴ First instalment = Rs. 5

Last instalment = Rs. 2,560.

Example 16 :

If the value of a machine is Rs. 1,60,000 and it depreciates at the rate of 10% per year what will be its value at the end of 5 years?

Let a be the initial value. Its value at the end of 1 st, 2 nd, 3 rd years is a(·1), a(·1)^2, a(·1)^3, a(·1)^4, ...

∴ The value of the machine at the end of

5th year = ar^5

= 1,60,000 (·1)^5

= Rs. 94,478·50

Example 17 :

A ball is dropped from a height of 80 metres. After striking the ground it rebounds to $\frac{4th}{5}$ of its previovs height. Find the height during its 6th rebound and the total distance travelled before coming to rest.

Initial height of the ball = 80 metres.

Height after the first rebound = $80 \times \frac{4}{5}$

,, ,, ,, 2nd rebound = $80 \times \left(\frac{4}{5}\right)^2$, etc.

∴ Height after the 6th rebound = $80 \times \left(\frac{4}{5}\right)^6$

= 20·97 metres

Total distance travelled before the ball coming to rest

$$= 80+2\times80\times\left(\frac{4}{5}\right)+2\times80\times\left(\frac{4}{5}\right)^2+\ldots\ldots\ldots\infty$$

$$= 80+\frac{2\times80\left(\frac{4}{5}\right)}{1-\frac{4}{5}}$$

$$= 80+2\times80\times4$$

$$= 720 \text{ metres.}$$

Example 18 :

Find the value of the recurring decimal $2\cdot\bar{5}$

$$2\cdot\bar{5} = 2\cdot555\ldots\ldots\ldots\ldots\ldots$$

$$= 2+\cdot5+\cdot05+\cdot005+\ldots\ldots\ldots\ldots\ldots\infty$$

$$= 2+\frac{0.5}{1-0.1}$$

$$= 2+\frac{5}{9} = \frac{23}{9}$$

Example 19 :

Insert three geometric means between 3 and $\frac{3}{16}$

Let the three GMs be x_1, x_2, x_3.

Then $3, x_1, x_2, x_3, \frac{3}{16}$ are in GP.

Let r be the common ratio.

$$\therefore \frac{3}{16} = 3\cdot r^4$$

$$r^4 = \frac{1}{16} \quad \text{(ie)} \quad r = \pm\frac{1}{2}$$

$r = -\frac{1}{2}$ is inadmissible since all the GMs should lie between 3 and $\frac{3}{16}$

$$\therefore \quad x_1 = 3 \times \frac{1}{2} = \frac{3}{2}$$

$$x_2 = 3 \times \frac{1}{2^2} = \frac{3}{4}$$

$$x_3 = 3 \times \frac{1}{2^3} = \frac{3}{8}$$

$\therefore$ The GMs are $\frac{3}{2}$, $\frac{3}{4}$ and $\frac{3}{8}$.

Example 20 :

k is the arithmetic mean of two quantities and p, q are the two geometric means between the same two quantities. Prove that $p^3+q^3=2kxy$.

Let the two quantities be x and y.

Since k is the AM between x and y

$$2k = x+y \qquad \ldots \quad (1)$$

Since p,q are the GMs between x and y.

x, p, q, y are in GP.

$\therefore$ $y = xr^3$ where r is the common ratio.

$$r = \left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$p = xr = x\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$q = xr^2 = x\left(\frac{y}{x}\right)^{\frac{2}{3}}$$